Q1. Find P.V. i^i.

Solution: we apply $z^c = e^{c\log z}$ to rewrite i^i as

$$i^i = e^{i\log i}.$$

$$\log i = \ln|i| + i\left(\frac{\pi}{2}+2n\pi\right) = \ln 1 + i\left(\frac{\pi}{2}+2n\pi\right) = (2n \quad)\pi i$$

where $n = 0, \pm1, \pm2, \ldots$

Now we write

$$i^i = \exp[i\log i] = \exp[i(2n+\tfrac{1}{2})\pi i] = \exp[-(2n+\tfrac{1}{2})\pi]$$

Hence

$$\text{P.V. } i^i = e^{i\,\text{Log}\, i} = \exp\left(-\frac{\pi}{2}\right)$$

Q2. Let $f(z) = \left(\frac{z}{\bar{z}}\right)^2$. Show that the limit of $f(z)$ as z tends to 0 does not exist.

Solution:

When z approaches to $(0,0)$ along the real axis, $z = x + i0$

$$\lim_{\substack{z\to 0\\ \text{along x-axis}}} f(z) = \lim_{x\to 0}\left(\frac{x}{\bar{x}}\right)^2 = \lim_{x\to 0}\left(\frac{x}{x}\right)^2 = 1.$$

when z approaches to $(0,0)$ along the imaginary axis, $z = 0 + iy$

$$\lim_{\substack{z\to 0\\ \text{along y-axis}}} f(z) = \lim_{y\to 0}\left(\frac{iy}{\bar{iy}}\right)^2 = \lim_{y\to 0}\left(\frac{iy}{-iy}\right)^2 = 1.$$

When z tends to orgin along the line $y=x$, we have $z=x+ix$

$$\lim_{\substack{z\to 0\\ \text{along } y=x}} f(z) = \lim_{x\to 0}\left(\frac{x+ix}{\overline{x+ix}}\right)^2 = \lim_{x\to 0}\left(\frac{x+ix}{x-ix}\right)^2 = \lim_{x\to 0}\left(\frac{x(1+i)}{x(1-i)}\right)^2 = -1$$

Since the limits are unique, it follows that $\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2$ does not exist.

Q3. Find $f'(z)$ when

(a) $f(z) = (2z^2+i)^5$ (b) $f(z) = \frac{z-1}{2z+1}$ $(z \neq -\frac{1}{2})$

Solution:

(a) $f'(z) = 5(2z^2+i)^4 \frac{d}{dz}(2z^2+i) = 5(2z^2+i)^4 \cdot 4z = 20z(2z^2+i)^4$

(b) $f'(z) = \frac{(2z+1)\frac{d}{dz}(z-1) - (z-1)\frac{d}{dz}(2z+1)}{(2z+1)^2} = \frac{(2z+1)-2(z-1)}{(2z+1)^2} = \frac{3}{(2z+1)^2}$

Q4 Show $f'(z)$ does not exist at any point z when

$f(z) = \mathrm{Re}\, z$.

Solution:

Let z be any point in complex plane.

$\Delta w = f(z+\Delta z) - f(z) = \mathrm{Re}(z+\Delta z) - \mathrm{Re}\, z = \mathrm{Re}(\Delta z)$

then $\frac{\Delta w}{\Delta z} = \frac{\mathrm{Re}(\Delta z)}{\Delta z}$.

If $\Delta z = (\Delta x, \Delta y)$ approaches $z=(x,y)$ horizontally through the points $(x+\Delta x, y)$, then $\frac{\Delta w}{\Delta z} = \frac{\Delta x}{\Delta x}$ $\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x} = 1$

If Δz approaches to (x, y) vertically through the points $(x, y+\Delta y)$,

then $\frac{\Delta w}{\Delta z} = \frac{0}{i\Delta y} = 0$,

Hence $\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ does not exist, it follows that $f'(z)$ does not exist. Since we assume z is arbitary point at the begining, then this conclusion holds for any point.

Q5. Use Cauchy-Riemann equation to show that $f'(z)$ does not exist at any point if

$$f(z) = e^{x} e^{-iy}$$

Solution:

$$f(z) = e^x e^{-iy} = e^x(\cos y - i\sin y) = e^x\cos y - ie^x\sin y$$

So $u(x,y) = e^x\cos y$, $v(x,y) = -e^x\sin y$.

Then $u_x(x,y) = e^x\cos y$ $\quad u_y(x,y) = -e^x\sin y$

$v_x(x,y) = -e^x\sin y$ $\quad v_y(x,y) = -e^x\cos y$

$u_x(x,y) = v_y(x,y) \Rightarrow e^x\cos y = -e^x\cos y \Rightarrow 2e^x\cos y = 0$

Since $e^x \neq 0$, $\forall x \in \mathbb{R}$, we must have $\cos y = 0$, $y = \frac{\pi}{2} + n\pi$, $n = 0, \pm1, \cdots$

$u_y(x,y) = -v_x(x,y) \Rightarrow e^x\sin y = -e^x\sin y \Rightarrow \sin y = 0$ $\quad y = n\pi$, $n = 0, \pm1, \cdots$

Since there are two different sets of values of y, so the C-R equation can not be satisfied.